Note on the Modulus of Continuity of Non-differentiable Functions given by Series with Small Gaps

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Abstract

In the note we sharpen the theorem of N.G. Gamkrelidze and show that under the conditions $n_k \sim \exp(c_1 k^{1-\alpha})$ as $k \to \infty$ and $n_{k+1}/n_k > 1 + c_2 k^{-\alpha}$ for some c_1, $c_2 > 0$ and $0 \leq \alpha \leq 1/4$ and let $f(x) = \sum_{k=1}^{\infty} n_k^{-1} \cos 2\pi n_k x$, then $f(x)$ is a continuous function on R and we have

$$\overline{\lim_{h \to 0}} \frac{f(x+h) - f(x)}{2\pi h \sqrt{\log 1/|h|)^{1/(1-\alpha)} (\log\log 1/|h|)}} = \sqrt{\frac{1}{c_1^{1/(1-\alpha)}(1-\alpha)}}, \ a.e.$$

and when $0 \leq \alpha < 1/4$,

$$\lim_{h \to 0} \text{meas} \left\{ x \in (0, 1);\ \frac{f(x+h) - f(x)}{2\pi h \sqrt{\frac{1}{2}(\log 1/|h|)^{1/(1-\alpha)}}} < c_1^{-1/2(1-\alpha)} y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{x^2}{2}\right) dx.$$

1. Introduction and Theorems

In this note, we will deepen the theorem of N.G. Gamkrelidze [3]. We show that it holds for more weak gaps than Hadamard's. In the following, we assume that $\{n_k\}$ is a sequence of positive integers such that

(1) $$n_k \sim \exp(c_1 k^{1-\alpha}) \quad \text{as} \longrightarrow \infty,$$

(2) $$n_{k+1}/n_k > 1 + c_2 k^{-\alpha} \quad \text{for some } c_1,\ c_2 > 0 \quad \text{and} \quad 0 \leq \alpha < 1,$$

where $a_k \sim b_k$ means $a_k/b_k \longrightarrow 1$ as $k \longrightarrow \infty$.
Then we can prove the following.

THEOREM 1. *Under the conditions* (1) *and* (2), *let* $f(x) = \sum_{k=1}^{\infty} n_k^{-1} \cos 2\pi n_k x$. *Then we have*

1) $$\overline{\lim_{h \to 0}} \frac{f(x+h) - f(x)}{2\pi h (\sqrt{\log 1/|h|})^{1/(1-\alpha)} (\log\log 1/|h|)} = \sqrt{\frac{1}{c_1^{1/(1-\alpha)}(1-\alpha)}}, \ a.e., \ if\ 0 \leq \alpha \leq 1/4,$$

2) $$\lim_{h\to 0} \text{meas}\left\{x\in(0,1);\ \frac{f(x+h)-f(x)}{2\pi h\sqrt{\frac{1}{2}\left(\log\frac{1}{|h|}\right)^{1/(1-\alpha)}}} < c_1^{-1/2(1-\alpha)}\,y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx,\ \text{if } 0\leqq\alpha<1/4.$$

COROLLARY. *Let b be an integer larger than unity and let $f(x)=\sum_{n=1}^{\infty} b^{-n}\cos 2\pi b^n x$. Then we have*

1) $$\overline{\lim_{h\to 0}}\ \frac{f(x+h)-f(x)}{2\pi h\sqrt{\left(\log_b\frac{1}{|h|}\right)\left(\log\log\frac{1}{|h|}\right)}} = 1\ a.e.,$$

2) $$\lim_{h\to 0} \text{meas}\left\{x\in(0,1);\ \frac{f(x+h)-f(x)}{2\pi h\sqrt{\frac{1}{2}\log_b\frac{1}{|h|}}} < y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx.$$

Next we show the similar result to hold for the Walsh-Paley system as the result for the Trigonometric system. We give the simple properties and definition in **4**. In **5**, the proof of *Theorem 2* is only to show for 1) and 2) is abbreviated as easy.

THEOREM 2. *Under the conditions* (1) *and* (2), *let $f(x)=\sum_{k=1}^{\infty} n_k^{-1}\,\psi_{n_k}(x)$. Then $f(x)$ is (W-) continuous and satisfied the following properties :*

1) $$\overline{\lim_{N\to\infty}}\ d_N(f,x)/\sqrt{2N^{\frac{1}{1-\alpha}}\log\log N} = \left(\frac{\log 2}{c_1}\right)^{\frac{1}{2(1-\alpha)}},\ a.e.,\ \text{if } 0\leqq\alpha\leqq 1/2,$$

2) $$\lim_{N\to\infty} \text{meas}\left\{x\in(0,1);\ d_N(f,x)/\sqrt{N^{\frac{1}{1-\alpha}}} < \left(\frac{\log 2}{c_1}\right)^{\frac{1}{2(1-\alpha)}} y\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{x^2}{2}}\,dx,\ \text{if } 0\leqq\alpha<1/2.$$

2. Preliminary

We use the following relations to prove our theorem ;

(3) $$\cos(\alpha+\beta)-\cos\alpha = -\sin\alpha\sin\beta - 2\cos\alpha\sin^2\frac{\beta}{2},$$

(4) $$\sin x = x - \frac{\theta x^2}{2}\quad \text{for some}\quad 0\leqq\theta\leqq 1,\quad \text{if}\quad |x|\leqq 1,$$

(5) for any $\varepsilon>0$, there exists an integer k_0 such that

$$(1-\varepsilon)e^{c_1 k^{1-\alpha}} < n_k < (1+\varepsilon)e^{c_1 k^{1-\alpha}}\quad \text{if}\quad k>k_0,$$

(6)
$$\left.\begin{array}{l} \dfrac{e^{c_1 N^{1-\alpha}}}{(1-\alpha)c_1}N^{\alpha}(1-o(1)) \\ \dfrac{e^{c_1}}{e^{c_1}-1}(1-o(1)) \end{array}\right\} < \sum_{k=1}^{N} n_k < \left\{\begin{array}{ll} \dfrac{e^{c_1 N^{1-\alpha}}}{(1-\alpha)c_2}N^{\alpha}(1+o(1)) & \text{if}\ \ 0<\alpha\leqq 1/4 \\ \dfrac{e^{c_1}}{e^{c_1}-1}e^{c_1 N}(1+o(1)) & \text{if}\ \ \alpha=0 \end{array}\right.\quad \text{as } N\longrightarrow\infty.$$

Since (3) and (4) is elementary and (5) is evident from (1), we show (6). For $0<\alpha\leqq 1/4$, using (2) and $1+x\geqq e^{(1-\alpha)x}$ if $0<x\leqq\delta$,

$$n_k/n_N<(1+c_2(N-1)^{-\alpha})^{-1}(1+c_2(N-2)^{-\alpha})^{-1}\cdots(1+c_2k^{-\alpha})^{-1}$$
$$<e^{-c_2(1-\alpha)\sum_{j=k}^{N-1}j^{-\alpha}} \quad \text{if} \quad k<N \quad \text{and} \quad c_2k^{-\alpha}\leqq\delta.$$

Hence, for sufficiently large N,

$$\sum_{N=1}^{N} n_k\leqq n_N\Big(\sum_{N=k_1+1}^{N}\exp(-c_2[N^{1-\alpha}-k^{1-\alpha}])+o(1)\Big)$$
$$<n_N\Big(e^{-c_2N^{1-\alpha}}\int_{k1}^{N+1}e^{c_2x^{1-\alpha}}dx+o(1)\Big)$$
$$\leqq n_NN^{\alpha}/(1-\alpha)c_2(1+o(1)).$$

From (5) and noting that for N' such that $(N')^{1-\alpha}\leqq(1-\varepsilon)N^{1-\alpha}<(N'+1)^{1-\alpha}$,

$$\sum_{k=1}^{N} n_k>\sum_{k=N'}^{N} n_k>(1-\varepsilon)\sum_{k=N'}^{N}e^{c_1k^{1-\alpha}}>\frac{1-\varepsilon}{c_1(1-\alpha)}\int_{c_1(1-\varepsilon)N^{1-\alpha}}^{c_1N^{1-\alpha}}e^t\Big(\frac{t}{c_1}\Big)^{\alpha/(1-\alpha)}dt$$
$$>\frac{1-\varepsilon}{c_1(1-\alpha)}e^{c_1N^{1-\alpha}}N^{\alpha}$$

$\alpha=0$ is shown as similar case. Thus we have proven (6).

3. Proof of the Theorem 1

It is clear that the $f(x)$ is continuous on R since the series is uniformly convergent. For any $h\neq 0$, there exists an integer N such that

$$\sum_{k=1}^{N} n_k\leqq(2\pi\,|\,h\,|)^{-1}<\sum_{k=1}^{N+1} n_k.$$

Thus for such h,

$$f(x+h)-f(x)=\Big(\sum_{k=1}^{N}+\sum_{k=N+1}^{\infty}\Big)n_k^{-1}\{\cos 2\pi n_k(x+h)-\cos 2\pi n_kx\}\equiv I_1+I_2, \text{ say.}$$

Then, from (5) we can estimate I_2 as follows,

$$|\,I_2\,|\leqq 2\sum_{k=1}^{\infty} n_k^{-1}=O\Big(\sum_{k=N+1}^{\infty}e^{-c_1k^{1-\alpha}}\Big)=O(e^{-c_1N^{1-\alpha}N^{\alpha}}).$$

To estimate I_1, we use (3) and (4),

$$I_1=-\sum_{k=1}^{N} n_k^{-1}\sin 2\pi n_kx\ \sin 2\pi n_kh-2\sum_{k=1}^{N} n_k^{-1}\cos 2\pi n_kx\ \sin^2\pi n_kh\equiv I_3+I_4, \text{ say.}$$

For I_4, $|\,I_4\,|\leqq 2\sum_{k=1}^{N} n_k^{-1}(\pi n_kh)^2=2\pi^2h^2\sum_{k=1}^{N} n_k\leqq\pi h^2\,|\,h\,|^{-1}=\pi\,|\,h\,|$.

Since $|\,2\pi n_kh\,|\leqq 2\pi n_k\,|\,h\,|\leqq\Big(\sum_{k=1}^{N} n_k\Big)^{-1}n_k\leqq 1$ if $k\leqq N$,

we have

$$\begin{aligned} I_3 &= -\sum_{k=1}^{N} n_k^{-1} \sin 2\pi n_k x \left\{ 2\pi n_k h - \frac{\theta_k}{2}(2\pi n_k h)^2 \right\} \\ &= -2\pi h \sum_{k=1}^{N} \sin 2\pi n_k x + 2\pi^2 h^2 \sum_{k=1}^{N} \theta_k n_k \sin 2\pi n_k x \\ &\equiv I_5 + I_6, \text{ say.} \end{aligned}$$

Therefore, $|I_6| \leqq 2\pi^2 h^2 \sum_{k=1}^{N} n_k \leqq \pi |h|$.

For simplicity, we use $\psi(u) = \sqrt{(u^{1/(1-\alpha)} \log u}$ if $u \geqq e$. Then it holds as follows

$I_2/2\pi h\psi(\log 1/2\pi|h|) = O(N^{2\alpha}/\psi(\log 1/2\pi|h|)) = O(N^{2\alpha-1/2}/\log\log 1/2\pi|h|) = o(1)$ and $(I_4 + I_6)/2\pi h\psi(\log 1/2\pi|h|) = O(1/\psi(\log 1/2\pi|h|)) = o(1)$ since $0 \leqq \alpha \leqq 1/4$ and (6). By S. Takahashi's theorem [4], we have

$$\overline{\lim_{N\to\infty}} \frac{\sum_{k=1}^{N} \sin 2\pi n_k x}{\sqrt{N \log\log N}} = 1, \ a.e.$$

Hence we can easily calculate ;

$$\lim_{h\to o} \frac{\sqrt{N \log\log N}}{\psi\left(\log \frac{1}{|h|}\right)} = \sqrt{\frac{1}{c_1^{1/(1-\alpha)}(1-\alpha)}}.$$

Combine this with the above fact and we obtain

$$\limsup_{h\to o} \frac{f(x+h)-f(x)}{2\pi h\psi(\log 1/|h|)} = \sqrt{\frac{1}{c_1^{1/(1-\alpha)}(1-\alpha)}}, \ a.e..$$

To prove 2) of the Theorem 1, we use again S. Takahashi's theorem [5] and we have

$$\lim_{h\to o} \text{meas} \left\{ x \in [0, 1) ; \frac{f(x+h)-f(x)}{2\pi h\sqrt{\frac{1}{2}\left(\log \frac{1}{|h|}\right)^{\frac{1}{1-\alpha}}}} < c_1^{-\frac{1}{2(1-\alpha)}} y \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx,$$

which completes the proof.

Proof of Corollary is obtained from *Theorem 1*, since $b^k = e^{k\log b}$.

4. Notations and Definitions for Walsh system

Let $\{\psi_n(x)\}$ be Walsh-Paley system on $[0, 1)$ and $\dot{+}$ be the group operation which N.J. Fine defined on the interval $[0, 1)$. In relation to the Walsh functions and the operation $\dot{+}$, it is known as follows ; for every nonnegative integer k and $x \in [0, 1)$, $\psi_k(x \dot{+} t) = \psi_k(x)\psi_k(t)$ holds except for finite points $t \in [0, 1)$. Here we define the derivative concept for the Walsh functions introduced by P.L. Butzer and H.J. Wagner.

Definition. ([1]) A real-valued function $f(x)$ is said to be dyadically differentiable at a point $x \in [0, 1)$ if $f(t)$ is defined at x and $x \dot{+} 2^{-n-1}$, $n = 0, 1, \cdots$, and if the sequence

$$d_N(f, x) \equiv \sum_{n=0}^{N-1} 2^{n-1}(f(x) - f(x \dot{+} 2^{-n-1})) \tag{7}$$

converges as $N \longrightarrow \infty$. In this case, we shall denote the limit of (6) by $df(x)$ and call it the dyadic derivative of f at x. It is known that every Walsh function is dyadically differentiable on $[0, 1)$ with $d\psi_k = k\psi_k$, $k=0, 1, \cdots$, and that if N and k are any non-negative integers and if k_0 satisfies $0 \leqq k_0 < 2^N$ and $k = k_0 (\mathrm{mod}\ 2^N)$ then

$$\sum_{k=0}^{N-1} 2^{n-1}(1-\psi_k(2^{-n-1})) = k_0. \tag{8}$$

5. Proof of the Theorem 2

For sufficient large integer N, there exists a $j=j(N)$ such that $n_j \leqq 2^N < n_{j+1}$. Since $n_k \sim \exp(c_1 k^{1-\alpha})$, we have $j \sim \left(\frac{1}{c_1}\log n_j\right)^{1/(1-\alpha)} \sim (c_1^{-1} N \log 2)^{1/(1-\alpha)}$. Now,

$$\begin{aligned} d_N(f, x) &= \sum_{n=0}^{N-1} 2^{n-1}(f(x) - f(x \dotplus 2^{-n-1})) \\ &= \sum_{n=0}^{N-1} 2^{n-1} \sum_{k=1}^{\infty} n_k^{-1}[\psi_{n_k}(x) - \psi_{n_k}(x \dotplus 2^{-n-1})] \\ &= \sum_{k=1}^{\infty} n_k^{-1}\psi_{n_k}(x)\left(\sum_{n=0}^{N-1} 2^{n-1}[1-\psi_{n_k}(2^{-n-1})]\right), \end{aligned}$$

noting $k' \equiv n_k (\mathrm{mod}\ 2^N)$,

$$\begin{aligned} &= \sum_{k=1}^{j} n_k^{-1} n_k \psi_{n_k}(x) + \sum_{k=j+1}^{\infty} n_k^{-1} k' \psi_{n_k}(x) \\ &\equiv I_1 + I_2, \text{ say.} \end{aligned}$$

we can easily estimate as follows, since $k' < 2^N$,

$$|I_2| \leqq \sum_{k=j+1}^{\infty} n_k^{-1} 2^N \leqq 2^N c_1^{-1} \int_j^{\infty} e^{-c_1 x^{1-\alpha}} dx \leqq 2^N c_1^{-1} e^{-j^{1-\alpha}} j^{\alpha} \leqq \text{Const. } j^{\alpha}.$$

For I_1, by A. Földes' theorem [2], we have

$$\overline{\lim_{j\to\infty}} \frac{I_1}{\sqrt{2j \operatorname{loglog} j}} = 1, \ a.e..$$

Therefore,

$$\overline{\lim_{N\to\infty}} \frac{d_N(f, x)}{\sqrt{2N^{\frac{1}{1-\alpha}} \log\log N}} = \overline{\lim_{N\to\infty}} \frac{I_1}{\sqrt{2j \log\log j}} \cdot \frac{\sqrt{2j \log\log j}}{\sqrt{2N^{\frac{1}{1-\alpha}} \log\log N}} = 1 \cdot \sqrt{\left(\frac{\log 2}{c_1}\right)^{\frac{1}{1-\alpha}}},$$

which completes the proof of Theorem 2, 1). To prove 2) of theorem 2, we can use again A. Földes' theorem [2] and we shall abbreviate it.

References

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