A Mixed Problem for an Infinite Elastic Medium Containing a Circular Inclusion and a Circular Hole*

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ABSTRACT

This paper presents the mixed boundary value problem for an infinite elastic medium which contains a circular hole and a circular inclusion with different elastic properties from the matrix. The inclusion is perfectly bonded to the matrix along the circumference, the circular hole is in contact with two rigid stamps.

The effect of a circular inclusion on the stress state along the rim of the circular hole is discussed.

The conformal mapping and the complex variable technique are used to obtain an analytical solution for the problem.

Numerical calculations are carried out for the stress distribution along the circumference of the circular hole due to the configuration and elastic properties of the matrix and the inclusion.

Introduction

Mixed boundary value problems for an infinite medium with two circular holes have been discussed by several authors [1–5]. However, the particular mixed problem for a double-connected region, i.e. the contact problem for an infinite elastic medium containing two circular boundaries has not been worked out. In recent years, the author [6] has been discussed the mixed boundary value problem for the infinite region bounded by two nonintersecting circular holes, one of which subjected to the uniform pressure and the other is in contact with two rigid stamps.

This paper continues the previous work and deals with the mixed boundary value problem for an infinite elastic medium (matrix) which contains a circular hole and a circular inclusion of different elastic material. The circular inclusion is bonded to the matrix along the circumference, the hole is in contact with two rigid stamps. On the basis of the two-dimensional theory of elasticity, the effect of an inclusion on the stress state along the rim of the circular hole is discussed. The conformal mapping and the complex variable approach are used throughout to convert this mixed boundary value problem to a Hilbert problem [7].

To clarify the effect of the configuration and the elastic properties of the matrix

and the inclusion on the stress distribution along the circumference of the circular hole, numerical calculations are carried out and the results are shown graphically.

Statement of the problem and basic equations

We employ a rectangular coordinate system, as shown in Fig.1. We consider an infinite elastic matrix containing two holes whose boundaries are non-intersecting circles $L_1$ and $L_2$, of radii $R_1$ and $R_2$, respectively, and whose centres are distant $d (> R_1 + R_2)$ apart. A circular disk of the different elastic material from the matrix is inserted into the circular hole $L_1$ in the matrix and bonded round. It will be assumed that this infinite elastic medium is compressed under diametrally opposite forces $P$ applied through two rigid stamps of equal width which are perfectly bonded to the matrix along the contact arcs $L_1$ and $L_2$ of $L$. Let the matrix and the inclusion be homogeneous and isotropic.

In the $z$-plane ($z = x + iy$), we denote the region inside the circle $L_2$ by $S_2$ and the remaining part of the plane by $S_1$. Moreover, in the region $S_2$, let $S_1$ be the interior of a circle $L_1$ and $S_2$ the exterior (Fig.2a).

As a very powerful method of solving the problem, we use conformal mapping. By [8], the function

$$z = \omega(\zeta) = \frac{\alpha \zeta + \delta}{\bar{\zeta} + \gamma}$$

maps the plane region $z = x + iy$ conformally on to the region $\zeta = \xi_0 + i \xi_2$ (Fig.2), where

$$\alpha = -d + R_1 \gamma, \quad \delta = -d \gamma + R_1, \quad \gamma = (1 + A)/(1 - A),$$

$$A^2 = |(d - R_1)^2 - R_2|^2/(d + R_1)^2 - R_2|^2|.$$
and $\Gamma_i (1 \leq i = r - (\delta + R_C)/(a + R_C) > 1)$ in the $\zeta$-plane, respectively. The regions $S'_i$ and $S''_i$ correspond to the regions $\Sigma'_i$ inside and the region $\Sigma''_i$ outside the circle $\Gamma_i$, respectively. Consequently, the region $S''_n$ is mapped onto the interior $\Sigma''_n$ of the circle $\Gamma_n$, the infinite region $S'_n$, consisting of the points outside two given circles $L_1$ and $L_2$, is mapped onto the ring $\Sigma''_n$ bounded by the two concentric circles $\Gamma_1$ and $\Gamma_2$.

Let complex potentials $\Phi_*(z)$, $\Psi_*(z)$ be defined in the $z$-plane and $\zeta$-plane are connected by the function (1), the potentials $\Phi_*(z)$, $\Psi_*(z)$ can be considered as the function of $\zeta$. New notation is used by designating $\Phi_*[\omega(\zeta)]$ as $\Phi(\zeta)$, and so on. Putting $\zeta = \rho e^{i\eta}$, the lines $\rho = \text{const.}$ and $\eta = \text{const.}$ form, in the $z$-plane, an orthogonal curvilinear network. Hence, when no body forces are present, the stress components $\sigma$, $\tau$, $\tau_\eta$ in the curvilinear coordinate system $(\rho, \eta)$ and the displacement components $u_\rho$, $u_\eta$ in the rectangular coordinate can be expressed in terms of the potentials $\Phi(\zeta)$, $\Psi(\zeta)$ of the complex variable $\zeta = \rho e^{i\eta}$:

$$\sigma_\rho + \sigma_\eta = 2[\Phi(\zeta) + \Phi(\overline{\zeta})],$$

$$\sigma_\rho + i\tau_\eta = \Phi(\zeta) + \Phi(\overline{\zeta}) - \frac{1}{\omega(\zeta)} \frac{\partial}{\partial \zeta} [\overline{\omega(\zeta)}\Phi'(\overline{\zeta}) + \overline{\omega'(\overline{\zeta})}\Psi(\overline{\zeta})]/|\zeta\omega'(\zeta)|,$$

$$2\mu(u'_\rho + iu'_\eta) = i\zeta\omega(\zeta)[\kappa\Phi(\zeta) - \Phi(\overline{\zeta}) + \frac{1}{\omega(\zeta)} \frac{\partial}{\partial \zeta} [\overline{\omega(\zeta)}\Phi'(\overline{\zeta}) + \overline{\omega'(\overline{\zeta})}\Psi(\overline{\zeta})]/|\zeta\omega'(\zeta)|],$$

where $\mu$ is the shear modulus, $\kappa = 3 - 4\nu$ for plane deformation and $\kappa = (3 - \nu)/(1 + \nu)$ for general plane stress, $\nu$ is Poisson's ratio, in addition, $u'_\rho = \partial u_\rho/\partial \eta$ and $u'_\eta = \partial u_\eta/\partial \eta$. Let $\Phi_*(z)$, $\Psi_*(z)$ be defined and holomorphic at every points of the region $S'_i$ except for possible singularities in $S''_i$. Then, the corresponding functions $\Phi_i(\zeta)$, $\Psi_i(\zeta)$ are defined for the region $\Sigma''_i$ of $\zeta$-plane and have singularities in the region $\Sigma''_i$.

We now extend the definition of $\Phi_i(\zeta)$ to the region $\Sigma''_i$ by putting

$$\omega'(\zeta)\Phi_i(\zeta) = -\omega'(\zeta)\overline{\Phi}_i(\overline{\zeta}) + (r'/\zeta)\omega(\zeta)\overline{\Phi}_i(\overline{r'}/\zeta) + (r'/\zeta')\overline{\omega'}(\overline{r'}/\zeta)\Psi_i(\overline{\zeta}), \zeta \in \Sigma''_i.$$ (5)

Also, we get

$$\omega'(\zeta)\Psi_i(\zeta) = (r'/\zeta)|\overline{\omega'}(r'/\zeta)\overline{\Phi}_i(\overline{r'}/\zeta) + \overline{\omega'}(r'/\zeta)\Phi_i(\overline{\zeta})| - \omega(r'/\zeta)\Phi_i(\overline{\zeta}), \zeta \in \Sigma''_i.$$ (6)

This formula expresses $\Psi_i(\zeta)$ for $\zeta$ in $\Sigma''_i$ terms of $\Phi_i(\zeta)$ which has been extended by (5). Hence, the components of stress and displacement in the matrix filling the region $S''_n$ may be expressed in terms of the single function $\Phi_i(\zeta)$,

$$\sigma_\rho + \sigma_\eta = 2[\Phi_i(\zeta) + \Phi_i(\overline{\zeta})]$$

$$\overline{\zeta}\omega'(\zeta)[\sigma_\rho + i\tau_\eta] = \overline{\zeta}\omega'(\zeta)\Phi_i(\overline{\zeta}) - (r'/\zeta)\omega'(r'/\zeta)\Phi_i(\overline{\zeta})$$

$$+ i\overline{\zeta}\omega'(\zeta) - (r'/\zeta)\omega'(r'/\zeta)\Phi_i(\overline{\zeta})| - \omega(\zeta) - \omega(r'/\zeta)|\Phi_i(\overline{\zeta}), \zeta \in \Sigma''_i,$$ (8)

$$2\mu(u'_\rho + iu'_\eta) = i\overline{\zeta}\omega'(\zeta)|\kappa\Phi_i(\zeta) + \Phi_i(\overline{r'}/\zeta)| - i\overline{\zeta}\omega'(\zeta)[\Phi_i(\overline{r'}/\zeta)/|r'\omega'(r'/\zeta)|]$$


where \( \mu_2 \) and \( \kappa_2 \) denote the elastic constants of the matrix and \( \Psi_2(\xi) \) is given by (6).

We suppose that the contact arcs \( L_{2,1} \) and \( L_{2,2} \) of \( L_2 \) subtend the angle \( 2\omega \) at the origin respectively (see Fig. 1). The union of the arcs \( L_{2,1} \), \( L_{2,2} \) will be denote by \( L_2^* \), so that \( L_2 = L_{2,1} + L_{2,2} \), and the remaining part of \( L_2 \) will be denote by \( L_{2,0} \). By (1), the arcs \( L_{2,1} \) and \( L_{2,2} \) correspond to the arcs \( \Gamma_{1,2} = \Gamma_{2,1} + \Gamma_{2,2} \) and \( \Gamma_{2,2} = \Gamma_{2,1} + \Gamma_{2,2} \) of the circle \( \Gamma_2 \), respectively. Let \( 2\omega_k \) be the central angle subtended by the arc \( \Gamma_{2,k} \), then the angle \( 2\omega \) and the angle \( 2\omega_k \) are connected by the formula

\[
\omega_{2,2} = \tan^{-1} \left[ \frac{R_2 R_2}{\gamma (1 - \gamma)} \sin \omega \right] \left[ (1 - \frac{R_2}{\gamma}) (\frac{R_2}{\gamma} - \gamma) \right]
\]

where the upper and lower signs apply to \( k = 1 \) and \( k = 2 \), respectively.

Let the complex potentials for the inclusion be \( \Phi_i(\xi), \Psi_i(\xi) \), then the corresponding functions \( \Phi_i(\xi), \Psi_i(\xi) \) have to be holomorphic in \( \Sigma_i \).

We denote by \( [\omega(\xi)\Phi_i(\xi)]^* \) and \( [\omega(\xi)\Phi_i(\xi)]^* \) the boundary values of \( \omega(\xi)\Phi_i(\xi) \) as \( \xi \) approaches at the point \( t \) of \( \Gamma_i \) from \( \Sigma_i \) and \( \Sigma_i^* \), respectively. By (8),(9), the boundary conditions of this problem can be written as

(a) On the arc \( L_{2,1}^* \), \( u_\sigma + i u_\omega = \text{const.} \);

\[
\kappa_2 [\omega(\xi)\Phi_i(\xi)]^* = 0, \quad t \in \Gamma_{2,1}.
\]

(b) On the arc \( L_{2,2}^* \), \( \sigma_y + i\sigma_\eta = 0 \);

\[
[\omega(\xi)\Phi_2(\xi)]^* - [\omega(\xi)\Phi_2(\xi)]^* = 0, \quad t \in \Gamma_{2,2}.
\]

(c) On the common circle \( L_1 \), i.e., on the circle \( \Gamma_1 \);

\[
\Phi_i(\sigma) + \Phi_i(\bar{\sigma}) - \bar{\sigma} \omega(\sigma)\Phi_i(\sigma) + \bar{\omega}(\bar{\sigma})\bar{\Phi}_i(\bar{\sigma})]/\| \sigma \omega(\sigma) \|
\]

\[
= \Phi_i(\sigma) + \Phi_i(\bar{\sigma}) - \bar{\sigma} \omega(\sigma)\Phi_i(\sigma) + \bar{\omega}(\bar{\sigma})\bar{\Phi}_i(\bar{\sigma})]/\| \sigma \omega(\sigma) \|,
\]

\[
(\mu_1/\mu_2)\kappa_2 [\sigma \omega(\sigma) - \bar{\sigma} \omega(\sigma)\Phi_i(\sigma) + \bar{\omega}(\bar{\sigma})\bar{\Phi}_i(\bar{\sigma})]/\| \sigma \omega(\sigma) \|
\]

\[
= \kappa_1 \sigma \omega(\sigma)\Phi_i(\sigma) - \sigma \omega(\sigma)\Phi_i(\sigma) + \bar{\sigma} \omega(\sigma)\Phi_i(\sigma) + \bar{\omega}(\bar{\sigma})\bar{\Phi}_i(\bar{\sigma})]/\| \sigma \omega(\sigma) \|
\]

where \( \sigma = e^{i\gamma} \) and \( \mu_1, \kappa_1 \) are elastic constants of the inclusion. In addition,

(d) Since the external forces applied to \( \Sigma_{2,1}^* \) have the resultant vector \( (-P, 0) \), we get
\[ i \int_{r^-} \left[ \begin{array}{c} \omega'(t) \Phi_2(t) \end{array} \right] - \left[ \begin{array}{c} \omega'(t) \Phi_1(t) \end{array} \right] dt = -P. \]  

Further, the condition of single-valuedness of displacements corresponding to these potentials must be satisfied.

**Complex Potentials**

The problem is reduced to construct the potential functions \( \Phi_1(\zeta), \Psi_1(\zeta) \), \( \Phi_i(\zeta) \), and \( \Psi_i(\zeta) \), which satisfy the boundary conditions (11) to (14) and the condition of single valuedness of displacements.

Condition (12) indicates that \( \omega'(\zeta)\Phi_i(\zeta) \) is continued analytically through the boundary arc \( \Gamma_r^- \). Condition (11) expresses the Hilbert boundary value problem for the function \( \omega'(\zeta)\Phi_(\zeta) \). In order to solve this Hilbert problem, we investigate the singularities \( \omega'(\zeta)\Phi_i(\zeta) \). From (4) we are that \( \omega'(\zeta) \) has a pole of the second order at \( \zeta = -\gamma \) \((1 < \gamma < r)\), corresponding to \( z = \infty \). Since the stress and rotation vanish at infinity in the matrix, \( \Phi_i(\zeta) \) is \( O[(\xi + \gamma)] \) near \( \zeta = -\gamma \). Thus \( \omega'(\zeta)\Phi_i(\zeta) = O(1) \) near the point \( \zeta = -\gamma \), in other words, \( \zeta = -\gamma \) will not be a pole of \( \omega'(\zeta)\Phi_i(\zeta) \). Further, \( \omega'(\zeta)\Phi_i(\zeta) \) may have the poles at the points \( \zeta = 0 \) and \( \zeta = \infty \), because the points of \( z \)-plane corresponding to \( \xi = 0 \) and \( \xi = \infty \) lie in the regions \( S_1 \) and \( S_2 \) occupying the hole and the inclusion, respectively. Consequently, the general solution of (11) is given by

\[ \omega'(\zeta)\Phi_i(\zeta) = X(\zeta) \sum_{n=1}^{\infty} b_n (\zeta/r)^n, \]  

where \( b_n \) are unknown coefficients and \( X(\zeta) \) is a particular solution of the Hilbert problem (11), is given by

\[ X(\zeta) = \left[ (\zeta - re^{-im\beta})(\zeta + re^{-im\beta}) \right]^{\frac{-\beta}{2}} \left[ (\zeta - re^{im\beta})(\zeta + re^{im\beta}) \right]^{\frac{+\beta}{2}}, \]  

where \( \beta = (\ln \kappa)/(2\pi) \).

Here \( X(\zeta) \) is understood as the branch, holomorphic in the plane cut along \( \Gamma_r^- \), such that \( \xi X(\zeta) \to 1 \) as \( |\zeta| \to \infty \).

The complex potentials \( \Phi_i(\zeta), \Psi_i(\zeta) \) for the elastic inclusion are to be holomorphic in the region \( \Sigma_i^- \). So that we can represent in the form,

\[ \omega'(\zeta)\Phi_i(\zeta) = \frac{1}{r} \sum_{\nu=0}^{\infty} g_{i\nu} \zeta^\nu, \]  

where \( g_{i\nu}, h_{i\nu} \) are unknown coefficients. The coefficients \( b_n, g_n, \) and \( h_n \) have to be determined so that they satisfy the conditions (13), (14), and (15) and the condition of single-valuedness of displacements.

Since the function \( \omega'(\zeta)\Phi_i(\zeta) \) is holomorphic in the region \( \Sigma_i^- \), except at the points \( \zeta = 0 \) and \( \zeta = \infty \), respectively, we have expansions.
\[ \omega' (\xi) \Phi_1 (\xi) = \frac{1}{r} \sum_{n=1}^{\infty} M_n (\xi/r)^n, \quad \xi \in \Sigma_i, \quad (19) \]
\[ \omega' (\xi) \Phi_2 (\xi) = \frac{1}{r} \sum_{n=1}^{\infty} N_n (\xi/r)^n, \quad \xi \in \Sigma_i, \quad (20) \]

where
\[ M_n = \sum_{k=0}^{\infty} P_n b_{k+1}, \quad N_n = \sum_{k=0}^{\infty} Q_n b_{k+1}, \quad (21) \]

\[ P_n, Q_n \text{ in (21) are the coefficients in the power series expansions of } X(\xi) \text{ in } \Sigma_i \text{ and } \Sigma_i \text{ respectively,} \]
\[ P_n = -\sum_{k=0}^{\infty} p_{n-k} p', \quad Q_n = \sum_{k=0}^{\infty} q_{n-k} q', \quad (22) \]

where
\[ p_n = e^{-z \beta_2 m}, \]
\[ p_n = 2 p_{n+1} \left[ \frac{\tilde{\eta} \gamma}{(2m)!} \cos \left( \frac{2m \omega_n}{3} \right) + (1 - \delta_{1,m}) \sum_{k=1}^{m-1} \frac{\tilde{\eta} \gamma}{k! (2m-k)!} \cos \left( \frac{2m \omega_n}{3} \right) \right] \quad (23) \]
\[ p_{n+1} = 2 p_n \left[ \frac{\tilde{\eta} \gamma}{(2m-1)!} \cos \left( \frac{2m \omega_n}{3} \right) \right] + (1 - \delta_{1,m}) \sum_{k=1}^{m-1} \frac{\tilde{\eta} \gamma}{k! (2m-k)!} \cos \left( \frac{2m \omega_n}{3} \right) \]

\[ \text{Here, } \gamma e^{z \beta_2} = (2j-1)/2 \pm i \beta, \quad q_n = q_{n-k} = 1, \quad \text{and the respective components } p', q, \text{ and } q' \]
\[ \text{may be obtained by replacing the } \omega_2 \text{ in (16) by } \omega_n, -\omega_n \text{ in turn.} \]

\[ \Phi_1 (\xi) \text{ is represented within } \Sigma_i \text{ by the Laurent series} \]
\[ \Phi_1 (\xi) = \frac{1}{r (\gamma-1)^2} \sum_{n=1}^{\infty} A_n (\xi/r)^n, \quad \xi \in \Sigma_i, \quad (24) \]

where
\[ A_n = \sum_{k=0}^{n} P_{n-k} d_{n+k}, \quad (25) \]
\[ d_n = r b_{n+2} + 2 \gamma b_{n+1} + \gamma' b_n. \quad (26) \]

Hence, substituting (20) and (25) into (9), the expression of \( \omega' (\xi) \Psi_1 (\xi) \) in Laurent series is given by
\[ \omega' (\xi) \Psi_1 (\xi) = \frac{1}{r} \sum_{n=-\infty}^{\infty} B_n (\xi/r)^n, \quad (27) \]
\[ \text{where} \]
\[ B_n = N_{n+2} - (n+1) \sum_{k=0}^{n} e_k A_{n+k} \quad (28) \]
\[ \text{where} \]
\[ e_0 = \frac{\alpha}{R (\gamma-1) r}, \quad e_k = (-1)^k \left( \frac{\gamma'}{r} \right)^{k-1}, \quad (k = 1, 2, \ldots). \]

Substituting (18), (19), (20), (24) and (27) into the continuity condition in (13) and (14), and comparing terms which contain the same power of \( \sigma \), we arrive at the relations
\[
2C_0 - D_0 - E_0 = 2A_0 - G_0 - H_0, \\
(\kappa_i - 1) C_0 + D_0 + E_0 = \mu_{ij} [(\kappa_i - 1) A_0 + G_0 + H_0],
\]

\[
C_n = [\lambda^n A_n - \lambda^n (G_n + H_n - A_n)], \\
\kappa_i C_n = \mu_{ij} [\kappa_i \lambda^n A_n - \lambda^n (G_n + H_n - A_n)], \\
C_n - D_n - E_n = [\lambda^n A_n - \lambda^n (G_n + H_n - A_n)], \\
(\kappa_i - 1) C_n - D_n - E_n = \mu_{ij} [(\kappa_i - 1) A_n + \lambda^n (G_n + H_n - A_n)], \\
(n = 1, 2, \ldots),
\]

where \( \lambda = 1/r, \mu_{ij} = \mu_1 / \mu_2 \), and

\[
C_n = \gamma g_n + (1 - \delta_{0,s}) 2\gamma g_{n-1} + (1 - \delta_{0,s} - \delta_{1,s}) g_{n-2}, \\
D_n = h_n + (1 - \delta_{0,s}) 2\gamma h_{n-1} + (1 - \delta_{0,s} - \delta_{1,s}) h_{n-2}, \\
E_n = \epsilon_i (n+1) C_{n+1} + (1 - \delta_{0,s}) \epsilon_j C_n + (1 - \delta_{0,s} - \delta_{1,s}) \epsilon_j c(n-1) C_n, \\
G_n = \epsilon_i (n+1) \lambda A_{n+1} + \epsilon_j n A_n + \epsilon_j (n-1) \lambda A_n, \\
H_n = \gamma^2 \lambda^3 B_{n+2} + 2 \gamma \lambda^2 B_{n+1} + B_n,
\]

here \( \epsilon_i = \frac{a}{R((\gamma') - 1)}, \epsilon_j = \frac{a \gamma + \delta}{R((\gamma') - 1)}, \epsilon_j = \frac{\gamma \delta}{R((\gamma') - 1)} \).

Moreover, from (15) we get

\[
\sum_{n=0}^{\infty} S_n b_n = \frac{2(1 + \kappa_2)}{2(1 + \kappa_2)} P, \\
\]

where

\[
\int_0^{2\pi} \frac{\cos(\beta \delta + n \eta)}{\sqrt{(\cos \eta + \cos \omega_\eta)(\cos \eta - \cos \omega_\eta)}} d\eta,
\]

here \( \delta_i = \ln \left[ \frac{\cos \frac{\eta - \omega_\eta}{2} \sin \frac{\omega_\eta - \eta}{2}}{\sin \frac{\eta + \omega_\eta}{2} \sin \frac{\eta + \omega_\eta}{2}} \right]. \)

Also the condition of single-valuedness of displacements gives

\[
M_{i-1} = 0, N_{i-1} = 0.
\]

Equations (29), (30), (32), and (35) are simultaneous equations for determining the coefficients \( b_i, g_n, \) and \( h_n \).

By eliminating \( g_n \) and \( h_n \) from (29) and (30), we obtain

\[
(\kappa_i - \mu_{ij} \kappa_i) \lambda^n A_n - (\kappa_i + \mu_{ij}) \lambda^n (G_n + H_n - A_n) = 0, \\
(\mu_{ij} - 1) \lambda^n (G_n + H_n - A_n) + (1 + \mu_{ij} \kappa_i) \lambda^n A_n = 0,
\]

\( (n = 1, 2, \ldots) \)

Hence, the equations (32), (35) and (36) are the required equations to determine the coefficients \( b_n \) of \( \Phi_n (\xi) \).
Numerical Examples

In order to clarify the effect of inclusion on the stress in the infinite matrix, numerical calculations were carried out for the following values:

\[ \kappa_1 = \kappa_2 = 2.0, \quad 2\omega = 60^\circ, \quad R_l/d = 0.3, \]
\[ R_l/d = 0.1 \text{ and } 0.5, \]
\[ \mu_2 = \mu_1/\mu_2 = 0.2, 0.5, 2.0. \]

In every case, the set of linear equations in (32), (35) and (36) is solved by the method of iteration and by taking the first 61 equations involving the first 61 unknowns \((b_1, b_2, b_3, \ldots, b_{61})\). The computations carried out revealed that the convergence of iteration is fairly good.

From (2) and (3), we obtain the stress on the circular \(L_2\).

The stress along the contact arc \(L_2' (L_{21} + L_{22}')\) are given by

\[
\sigma_\theta = \frac{1}{R_1(y-1)} \frac{e^{\beta_1} \sin \omega_1}{\sqrt{\cos \omega_1 + \cos \omega_2}} \sum_{\infty} d \cos \beta \delta_1 + (n-1) \eta, \quad \tau_{\theta \gamma} = \frac{1}{R_1(y-1)} \frac{e^{\beta_1} \sin \omega_1}{\sqrt{\cos \omega_1 + \cos \omega_2}} \sum_{\infty} d \sin \beta \delta_1 + (n-1) \eta, \quad (37)
\]

where the upper sign is chosen for the stress on \(L_{21}'\), while the lower sign corresponds to the stress on \(L_{22}'\), and \(d_1, \delta_1\) are determined by (26) and (34), respectively.

Further, \(-\omega_{21} < \eta < \omega_{21}\) for \(L_{21}'\), \(\pi - \omega_{21} < \eta < \pi + \omega_{21}\) for \(L_{22}'\).

The stress on the traction-free part \(L_2'' (L_{21}'' + L_{22}'')\) are given by

\[
\sigma_\gamma = \frac{4}{R_1(y-1)} \frac{e^{\beta_1} \sin \omega_1}{\sqrt{\cos \omega_1 + \cos \omega_2}} \sum_{\infty} d \sin \beta \delta_1 + (n-1) \eta, \quad \sigma_\theta = \tau_{\theta \gamma} = 0, \quad (38)
\]

where \(\delta_1 = \ln \left[ \left( \frac{\cos \frac{\eta - \omega_2}{2}}{\sin \frac{\eta - \omega_1}{2}} \right) \left( \frac{\cos \frac{\eta + \omega_2}{2}}{\sin \frac{\eta + \omega_1}{2}} \right) \right] \), \(\eta\) is given by

\[
\eta = \tan^{-1} \left[ \frac{R_1 R_2 (1 - \gamma) \sin \theta}{a \delta - (a \gamma + \delta) R_1 \cos \theta + R_2 \gamma} \right]. \quad (40)
\]

Fig. 3 and 4 show respectively the distribution of \(\sigma_\theta\) and \(\tau_{\theta \gamma}\) along the contact arc \(L_2'\). Fig. 5 shows the distribution of \(\sigma_\gamma\) on \(L_{22}'\), where \(q = P/(2R_1 \omega)\). In the Fig. 3 and
4. \((\sigma_p)_{21}\) indicates the stress \(\sigma_p\) on \(L_2\), and so on.

In the case of \(R_i/d=0.1\) the stress distribution along \(L_2\) are not affected excessively by the rigidity modulus of inclusion. But for the case \(R_i/d=0.5\) the shear modulus of inclusion has a great influence on the stress distribution around \(L_2\). Further, we see that in the case of \(\mu_1/\mu_2=0.5\) the stress distribution is about the same as the result given in [6].

![Fig. 3 Distridutions of \(\sigma_p\) along contact part of \(L_2\)](image)

![Fig. 4 Distribution of \(\tau_{\rho\eta}\) along contact part of \(L_2\)](image)
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