A Note on Applications of the Balanced Conditions

By

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Let \( \mathcal{M} = \{ M_1, \ldots, M_n \} \) be a family of serial left \( A \)-modules. Denote by \( S \) a set of the whole pairs \( (S, M_i) \) where \( M_i \) is a member in \( \mathcal{M} \) and \( S \) is a subquotient of \( M_i \). We shall say that \( \mathcal{M} \) satisfies the CBH-condition, if the following condition holds:

For elements \((S, M_i)\) and \((T, M_j)\) in \( S \) with \( S \sim T \), there exists a sequence of elements in \((S, M_i) = (S_{0}, M_{o}), (S_{1}, M_{1}), \ldots, (S_{t}, M_{t}) = (T, M_{j})\) such that an isomorphism between \( S_{p-1} \) and \( S_{p} \) can be induced by a homomorphism from \( M_{i_{p-1}} \) to \( M_{i_{p}} \) or inverse for each \( 1 \leq p \leq t \). In [3] we have characterized the balancedness of modules by the CBH-condition plus the constructibility and the other condition. (As for the definition of the constructibility see [3], [5] and [9].) In this note we shall make clear that this characterization is effective by means of applying it to solve some problems.

Section I will be devoted to determine completely such an algebra over an algebraically closed field as has a faithful balanced serial module which is projective or injective. And it will be known that only two types of rings are possible as such algebras. Perhaps it will be of interest how the constructibility is applied in the argument. The author [4] has shown that a QF-3 maximal quotient ring \( A \) whose minimal faithful left module is a direct sum of serial modules must be a generalized uniserial ring. The proof given there is based on investigation about the endomorphism rings of modules over a generalized uniserial ring. In Section II, by using the CBH-condition, we shall give another proof for this fact when \( A \) is a finite dimensional algebra over an algebraically closed field.

Throughout this paper, \( A \) denotes a ring with identity, \( N \) its Jacobson radical. An \( A \)-module means an unital module. For a left \( A \)-module \( M \), by \( \text{Rad}(M) \) we denote the radical of \( M \), the intersection of all maximal submodules of \( M \), and by \( \text{Soc}(M) \) the socle of \( M \), the sum of all minimal submodules of \( M \). If \( A/N \) is Artinian, then \( \text{Rad}(M) \) is equal to \( NM \). By \( T(M) \) we shall denote \( M/\text{Rad}(M) \). In case \( M \) is a serial module, \( \text{Rad}(M) \) and \( \text{Soc}(M) \) coincide with the unique maximal submodule and the unique minimal submodule of \( M \), respectively. If \( M \) has a composition series, denote by \( c(M) \) its length.

§ 1. Algebras which have balanced serial modules.

The purpose of this section is to determine completely algebras having faithful
balanced serial modules which are projective or injective. It will be clarified that such an algebra must be generalized uniserial. As for its proof, the use of the constructibility of balanced modules is essential one.

Lemma 1.1. Let \( A \) be a finite dimensional algebra over a perfect field. If a left \( A \)-module \( M \) is balanced serial, then \( \mathcal{T}(M) \cong \text{soc}(M) \).

Proof. We shall use induction on the length \( c(M) \) of \( M \). When \( c(M) = 1 \), it is trivial. Let \( c(M) \geq 2 \). Let \( \sigma(M) \) be of the maximal length among all submodules of \( M \) which are images of endomorphisms of \( M \). Then \( \sigma(M) \) is balanced, by [5, Theorem 1], and \( c(\sigma(M)) \geq 1 \) by [9, Theorem 2.6] or [5, Proposition 6]. Hence we have \( \mathcal{T}(\sigma(M)) \cong \text{soc}(\sigma(M)) \) by induction. Clearly \( \mathcal{T}(M) \cong \mathcal{T}(\sigma(M)) \) and \( \text{soc}(M) \cong \text{soc}(\sigma(M)) \). Thus we obtain \( \mathcal{T}(M) \cong \text{soc}(M) \).

Let \( M \) be a balanced serial left \( A \)-module with \( c(M) = n \) which is injective. Then, notice that a left \( A \)-module having the socle isomorphic to \( \text{soc}(M) \) is necessarily serial, and such a module is determined up to isomorphisms by the length. Let \( I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n = M \) be the whole of all non-zero submodules of \( M \) which are the images of endomorphisms of \( M \). Then, from the above fact it follows that \( c(I_1), c(I_2), \ldots, c(I_n) \) is equal differential. By Lemma 1.1, \( c(I_i) = 1 \), so if \( d \) is common difference, this equal differential sequence is as follows.

\[
1, 1 + d, 1 + 2d, \ldots, 1 + kd = n.
\]

Let \( K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \) be the whole of all proper submodules of \( M \) which are the kernels of endomorphisms of \( M \). Then, as easily seen, \( c(K_1), c(K_2), \ldots, c(K_n) \) is equal to \( d, 2d, \ldots, kd \).

Now, by easy length calculation, it can be seen that the whole of submodules of \( M \) whose length is one of

\[
1, 1 + d, \ldots, 1 + kd, d, 2d, \ldots, kd,
\]

that is, the whole of such submodules is a constructible set in the sense of [3]. Applying the constructibility of balanced modules [9, Theorem 3.2], it follows that any natural number less than \( n = 1 + kd \) must appear in \( 1, 1 + d, \ldots, 1 + kd, d, 2d, \ldots, kd \).

From this we conclude that \( d = 1 \) or \( d = 2 \). Let \( \sigma \) be an endomorphism of \( M \) such that \( \sigma(M) \) is the maximal image of endomorphism. Then, when \( d = 1 \), \( M \supseteq \sigma(M) \supseteq \sigma^2(M) \supseteq \cdots \supseteq \sigma^n(M) = 0 \) is the whole of all submodules of \( M \). And, when \( d = 2 \), \( M \supseteq \sigma(M) \supseteq \sigma^2(M) \supseteq \cdots \supseteq \sigma^n(M) \) is the whole of all non-zero submodules of \( M \) which are the images of endomorphisms of \( M \). Clearly, the same conclusion can be led, if we assume that \( M \) is projective instead of "injective".

Lemma 1.2. Let \( A \) be a finite dimensional algebra over an algebraically closed field \( K \), and \( M \) a serial left \( A \)-module. If \( M \supseteq \sigma_1(M) \supseteq \sigma_2(M) \supseteq \cdots \supseteq \sigma_n(M) \supseteq 0 \) is the whole
of all image submodules of $M$, where each $\sigma_i$ is an endomorphism of $M$, then $1, \sigma_1, \sigma_2, \ldots, \sigma_n$ form a $K$-basis of the endomorphism ring $B$ of $M$.

Proof. It will be proved by induction on the length of $M$. For each $1 \leq i \leq n$, there exists an endomorphism $\tau_i$ of $\sigma_i(M)$ such that $\sigma_i = \tau_i \sigma_i$ by [9, Lemma 2.1]. Obviously $\sigma_1(M) \supseteq \sigma_2(M) \supseteq \cdots \supseteq \sigma_n(M) = 0$ is the whole of all image submodules of $\sigma_i(M)$. Thus, by induction, $1, \tau_1, \ldots, \tau_n$ form a $K$-basis of the endomorphism ring $C$ of $\sigma_i(M)$. Let $\sigma$ be a given endomorphism of $M$. Since $B$ is a local algebra over an algebraically closed field $K$, $B = K + J(B)$, where $J(B)$ denotes the Jacobson radical of $B$. Hence there exist $k \in K$ and $\sigma \in J(B)$ such that $\sigma = k + \sigma'$. Since $\sigma'(M) \supseteq M$, we have $\sigma'(M) \subseteq \sigma_i(M)$, thus there exists an endomorphism $\tau$ of $\sigma_i(M)$ with $\sigma' = \tau \sigma$. Since $1, \tau_1, \ldots, \tau_n$ form a basis of $C$, we can take $k \in K$, $1 \leq i \leq n$, with $\tau = k_1 \cdot 1 + k_2 \cdot \tau_1 + \cdots + k_n \cdot \tau_n$. Then,

$$\sigma = k + \sigma' = k + \tau \sigma = k + \left( k_1 \cdot 1 + k_2 \cdot \tau_1 + \cdots + k_n \cdot \tau_n \right) \sigma_i$$

$$= k + k_1 \cdot 1 \sigma_i + k_2 \cdot \tau_1 \sigma_i + \cdots + k_n \cdot \tau_n \sigma_i$$

$$= k \cdot 1 + k_1 \cdot \sigma_i + k_2 \cdot \sigma_i + \cdots + k_n \cdot \sigma_i$$

This implies that $1, \sigma_1, \sigma_2, \ldots, \sigma_n$ form a $K$-basis of $B$.

Now we shall state the main theorem.

Theorem 1.3. For a finite dimensional algebra $A$ over an algebraically closed field $K$, the following statements are equivalent.

1) $A$ has a faithful balanced serial left module which is projective.

2) $A$ has a faithful balanced serial left module which is injective.

3) $A$ is either

a) a ring Morita equivalent to a local uniserial ring.

b) a generalized uniserial ring with the admissible sequence $2m+1, 2m$.

Proof. Assume 1) or 3), and let $\mathcal{M}$ be a faithful balanced serial left $A$-module such that $\sigma(M)$ is maximal among all image submodules of $M$. Then, by the above note, $c(\text{Ker}(\sigma)) = 1$ or $c(\text{Ker}(\sigma)) = 2$, and $M \supseteq \sigma(M) \supseteq \sigma^2(M) \supseteq \cdots$ is the whole of all image submodules of $M$. It follows that $\text{End}_A(M) = B = K \cdot 1 + K \cdot \sigma + K \cdot \sigma^2 + \cdots$ from Lemma 4. This means that $\text{End}_A(M) = B$ is a local uniserial algebra over $K$.

Now, let assume that $A$ is basic. Then, notice that $\dim A_S = 1$ for every simple $A$-module $S$. When $c(\text{Ker}(\sigma)) = 1, \dim A = \dim B = n$. Take an element $x \in M \setminus N_M$, then obviously $xB = M$. It follows that $M = xB \cong B_{x}$ considering the $K$-dimension. Since $\mathcal{M}$ is balanced, we conclude that $A \cong \text{End}(M) \cong \text{End}(B) = B$. Thus in this case,
A is a local uniseriaL Secondly, when $\mu$ is $\frac{n}{2m-1}$. Take elements $x \in M \setminus NM$ and $\gamma \in N \setminus N^2M$. Then $M_{\gamma} = xB_{\gamma} + \gamma B_{\gamma}$. Moreover, by noting the $K$-dimension, it can be seen that $xB_{\gamma} = B_{\gamma}$ and $\gamma B_{\gamma} = J$, where $J$ is the Jacobson radical of $B$. Therefore we obtain that $A \cong \text{End}(M_{\gamma}) \cong \text{End}(\nu B_{\gamma}) \cong \text{End}(B_{\gamma} + J)$. By means of the method used in [4], it can be shown that $\text{End}(B_{\gamma} + J)$ is a generalized uniseriaL ring with the admissible sequence $2m-1, 2m-2$.

Conversely, if $A$ is a ring stated in either of a) or b) of 3), clearly a unique minimal faithful left $A$-module is a faithful balanced serial module which is projective and injective (see [1] or [3]). This completes the proof.

Remark. In the proof of Theorem 5, if $c(\text{Ker}(\sigma)) = 2$, $c(M)$ is odd. So, if the length of $M$ in the proof is even, then $A$ is necessarily a ring Morita equivalent to local uniseriaL ring.

Remark. Since a generalized uniseriaL ring has left and right symmetric properties, in Theorem 1.3 we may add the right-hand version of a) or b) as an equivalent statement.

§ 2. An application of the CBH-condition.

We begin with a fact which can be led by the CBH-condition.

Proposition 2.1. Let $A$ be a finite dimensional algebra over a perfect field, and $M_1$, $\ldots$, $M_n$ serial left $A$-modules. Let $S$ be a subquotient of some $M_i$ with $c(S) \geq 2$. Assume that there exists a subquotient $T$ of some $M_i$ with $S \neq T$ and $S/Soc(S) = T/Soc(T)$ (resp. $\text{Rad}(S) = \text{Rad}(T)$). If $M_1 \oplus \cdots \oplus M_n$ is balanced, then $S$ is isomorphic to a submodule (resp. factor module) of some $M_i$.

Proof. Let $c(S) = c(T) = h + 1$ and $S/Soc(S) = N^2M_i/N^{2+h}M_i$, $T/Soc(T) = N^1M_i/N^{1+h}M_i$. Since $M_1 \oplus \cdots \oplus M_n$ is balanced, by the CBH-condition we have that there exists a sequence of subquotients

$$S/Soc(S) = N^2M_i/N^{2+h}M_i \leftarrow N^3M_i/N^{3+h}M_i \leftarrow \cdots \leftarrow N^pM_i/N^{p+h}M_i,$$

and $T/Soc(T) = N^1M_i/N^{1+h}M_i \leftarrow T/Soc(T)$

satisfying the following condition: There exists an $A$-homomorphism $\sigma: M_{ir} \rightarrow M_{ir+1}$ with $\sigma_r(N^rM_i) = N^{r+1}M_{ir+1}$ or an $A$-homomorphism $\tau: M_{ir+1} \rightarrow M_i$ with $\tau(N^{ir+1}M_i) = N^{ir}M_{ir}$ for each $0 \leq r \leq p - 1$. Suppose that $S/Soc(S)$ is isomorphic to no submodule of any $M_i$. Then, $N^{ir}M_i/N^{ir+h}M_i$, which is isomorphic to $S/Soc(S)$ never be a submodule of $M_{ir}$, that is, $N^{ir+h}M_i = 0$ for each $0 \leq r \leq p$. Thus, $\sigma_r$ or $\tau_r$ induces an isomorphism between $N^{ir}M_i/N^{ir+h}M_i$ and $N^{ir}M_{ir+1}/N^{ir+1+h}M_{ir}$ for each $0 \leq r \leq p - 1$. As $S = N^qM_i/N^{q+h}M_i$ and $T = N^qM_i/N^{q+h}M_i$ we obt.
ain \( S \simeq T \). This is a contradiction. The other part can be proved by quite same argument.

**Proposition 2.2.** Let \( A \) be a finite dimensional algebra over an algebraically closed field \( K \). Let \( M_1, \ldots, M_n \) be serial left \( A \)-module having the next properties.

1) \( M_1 \oplus \cdots \oplus M_n \) is faithful.
2) For subquotients \( S \) of some \( M_i \) and \( T \) of some \( M_j \), if \( T(S) \simeq T(T) \), then \( S \simeq T \).

Then \( A \) is a left serial algebra.

**Proof.** We may assume that \( A \) is basic. Then, notice that \( \dim_K W = 1 \) for every simple \( A \)-module \( W \). Now we want to show that \( A \) is a serial left \( A \)-module for a given primitive idempotent \( e \). Since \( M_1 \oplus \cdots \oplus M_n \) is faithful, there exists a subquotient \( U \) of some \( M_i \) with \( T(U) \simeq T(Ae) \). Let \( V \) be of the maximal length among such subquotients. Take an element \( x = ax \in V \setminus NV \). Consider a short exact sequence

\[
0 \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} V \longrightarrow 0,
\]

where \( f \) is an \( A \)-homomorphism defined by \( f(a) = ax \). Then, obviously \( \ker(f) \supseteq A(1 - e) \).

Now suppose that \( \ker(f) \supseteq A(1 - e) \). Then we can choose a non-zero element \( b \in Ae \) such that \( bx = 0 \). Let \( N^0M_i \supseteq N^1M_i \supseteq \cdots \supseteq N^mM_i \), \( 0 \leq t_1 < t_2 < \cdots < t_m \), be the whole of all submodules of \( M_i \) whose top is isomorphic to \( T(Ae) \). Take a given element \( y \in M_i \). Clearly \( ey \in N^1M_i \). The assumption assures that there exists an epimorphism \( g : V \longrightarrow N^1M_i \). Obviously \( g(x) \in N^1M_i/N^2M_i \). So we can take \( k \in K \) and \( y' \in N^2M_i \) with \( ey = kg(x) + y' \). Then, considering the fact \( bx = 0 \), we have

\[
by = bey = b(\chi g(x) + y') = kg(bx) + by' = by',
\]

where \( y' \in N^1M_i \) and \( ey' \in N^2M_i \). Continuing this process, we conclude \( by = 0 \). As \( M_i \) is any member \( M_i, \ldots, M_n \), it follows that \( b(M_i \oplus \cdots \oplus M_n) = 0 \). This contradicts the assumption that \( M_i \oplus \cdots \oplus M_n \) is faithful. So we obtain that \( Ae \simeq A/A(1 - e) = A/\ker(f) \simeq V \) is serial, completing the proof.

Now, by using the above propositions, we shall show the assertion: \( A \) is a generalized uniserial algebra, if \( A \) is a QF-3 maximal quotient algebra over an algebraically closed field having a minimal faithful left module \( Ae_1 \oplus \cdots \oplus Ae_n \) where each \( Ae_i \) is serial.

First, note that the condition on \( A \) is right and left symmetric. For the aid of Proposition 2.2, it suffices to show a contradiction on the hypothesis that there exist subquotients \( N^{p_i}e_i/N^{q_i}e_i \) of some \( Ae_i \) and \( N^{r_i}e_i \) such that \( N^{p_i}e_i/N^{q_i}e_i \simeq N^{r_i}e_i/N^{s_i}e_i \) and \( N^{q_i}e_i/N^{s_i}e_i \neq N^{p_i}e_i/N^{r_i}e_i \). Since \( Ae_1 \oplus \cdots \oplus Ae_n \) is balanced by \( [8] \), applying Proposition 1, it follows that \( N^{q_i}e_i/N^{s_i}e_i \simeq N^{r_i}e_i/N^{t_i}e_i \) are embeddable into an injective module \( A \). By the
length, a left $A$-module having the socle isomorphic to $S(Ae_i)$ is determined up to isomorphisms, so $Ae_i / N^{s-t}e_i \cong N^{s-t}e_i / N^{s+t}e_i$ if $s \leq t$. Therefore exists an epimorphism $Ae_i \rightarrow N^{s-t}e_i \rightarrow 0$, and this epimorphism induces an isomorphism between $N^{s-t}e_i / N^{s+t}e_i$ and $N^{s-t}e_i$, a contradiction.

Example. Let $A$ be a subalgebra of the full matrix ring of degree 4 over a field $K$ with a $K$-basis

$$c_{11} + c_{44} = e_1, \quad c_{22} + c_{33} = e_2, \quad c_{11}, \quad c_{22}, \quad c_{33}, \quad c_{44}.$$ 

where $c_{ij}$ denotes a matrix in whose $(i, j)$-position is 1 and 0 elsewhere. Then $A$ is a QF-3 algebra with a minimal faithful left module $Ae_1$. Moreover $Ae_1$ is a serial module. On the contrary, $Ae_2$ is not serial. For $Ne_2 / N^2e_2 \cong T(Ae_1) \oplus T(Ae_2)$ is not simple. From this example we see that the hypothesis "maximal quotient" cannot be removed as for the above assertion.

References